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MODES IN LINED WEDGE-SHAPED DUCTS

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The computation of sound fields in wedge-shaped spaces with an absorbing boundary (the seabed) is a classical problem of underwater acoustics, covered by a large number of publications. All known solutions are approximations which are restricted to very small wedge angles ϑ_0, typically less than 3°. In underwater acoustics is is further assumed that $k_0 r \gg 1$. The background of the present paper is the performance of lined conical duct sections in silencers. There the wedge angle can attain values around 45°, and the assumption $k_0 r \gg 1$ cannot be made. The absorber of the lined boundary here is supposed to be locally reacting (for reasons of simplicity); it can be characterized by a normalized surface admittance G_0. The problems of the analysis arise from the fact, that the fundamental field solutions (modes) can no longer be separated in the cylindrical co-ordinates r, ϑ if a boundary is absorbing. This paper describes analytical solutions for the construction of modes in lined wedge-shaped ducts; they can be applied for wedge angles up to about 15° (a subsequent paper will describe a method for angles up to about 45° but only moderate $k_0 r$ values). In the solutions, use is made of "fictitious modes", which satisfy the boundary conditions and solve a part of the wave equation. They must be completed by a "modal rest" to satisfy approximately the full wave equation. In the first solution, the rest is synthesized by fictitious modes; in the second solution, a separate function is introduced for the rest. Modes for typical underwater acoustics conditions will arise as side products.

1. INTRODUCTION

A practical task forms the background of the present analysis: the performance of a wedge-shaped transition between two silencer sections having different free duct widths. Solutions are known when the transition either has rigid walls, or if the lined walls are in the shape of steps (see, for example, reference [1]); however, steps in the duct will unduly increase the stationary flow resistance of the silencer, and it is believed that conical lined transitions can be designed as mode filters. Useful instruments for the description of sound fields in lined wedge-shaped ducts would be modes, i.e., elementary functions which solve the wave equation, satisfy the boundary conditions and are orthogonal over the cross-section. With such modes the field fitting at the entrance and exit of the conical section could easily be formulated, and the sound field of a source in the wedge could be synthesized.

The sound field in a wedge-shaped space is an important topic in underwater acoustics; correspondingly high is the number of publications about the problem;

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a complete reference of the literature would be outside the scope of this paper; only a short survey shall be given. When the seabed is assumed to be sound penetrable then the underwater acoustic task is similar to our task. However, most of the papers dealing with such conditions cannot be applied to our problem. Many papers apply the parabolic equation method (PE method). The parabolic equation is an approximated wave equation in cylindrical co-ordinates which—with the assumptions of a small wedge angle ϑ_0 and of large $k_0 r$ values—can be written as a product of two differential operators, for which approximate solutions are developed. The PE methods cannot be applied here, both because the wedge angle there must be really small (below about 3°) and $k_0 r \gg 1$ generally is not true here. Arnold/Felsen [2] and Topuz and Felsen [3] have derived the analysis of "intrinsic modes" in a wedge-shaped ocean environment (soft upper boundary, bulk reacting lower boundary), which are elementary solutions of the wave equation and boundary conditions suited for the synthesis of general fields (similar constrictions in [4]). Although the authors claim that their analysis is exact, the numerical implementation falls back on the conditions of small wedge angle and large $k_0 r$. Possibly the main difference between PE methods and the present method is that the PE methods approximate the wave equation by a product of differential operators, whereas here it is divided into a sum of two operators.

One can see four differences between typical underwater acoustic conditions and our task: The upper boundary of the wedge in the ocean is soft (pressure release), whereas we are mainly interested in fields symmetrical relative to the central plane of a two-sided wedge, so that the plane of symmetry (at $\vartheta = 0$) can be assumed as a rigid boundary in a one-sided wedge; this difference would not produce serious difficulties, although its influence on the sound field is important. The absorbing boundary of the seabed is a bulk reacting layer, whereas the absorbing boundary here is locally reacting. This difference would make necessary a reformulating of underwater acoustic methods, because seabeds are "fast" media (sound speed higher than in water) whereas porous absorber materials for air-borne sound are "slow"materials. That difference would become important, because the numerical evaluations in underwater acoustics apply pass integrations. Even more serious are the differences in the magnitude of the wedge angle ϑ_0 and the range of $k_0 r$.

Nevertheless one can learn principal features from the underwater acoustics literature: the wave equation in cylindrical co-ordinates no longer separates although the boundary surfaces are co-ordinate surfaces; on must be satisfied to solve the wave equation approximately; analytical formulations of modes oftern have numerical procedures embedded; a wave which, for example, propagates towards the apex of the wedge will generate continuous reflections; it is reasonable to formulate modes such that these reflections are included.

2. FORMULATION OF THE TASK

Our object is depicted in Figure 1. Normally the configuration is as in Figure 1(a): both boundaries of the wedge area are lined with identical absorbers. Basic field solutions in the wedge area are either symmetrical with respect to $\vartheta = 0$

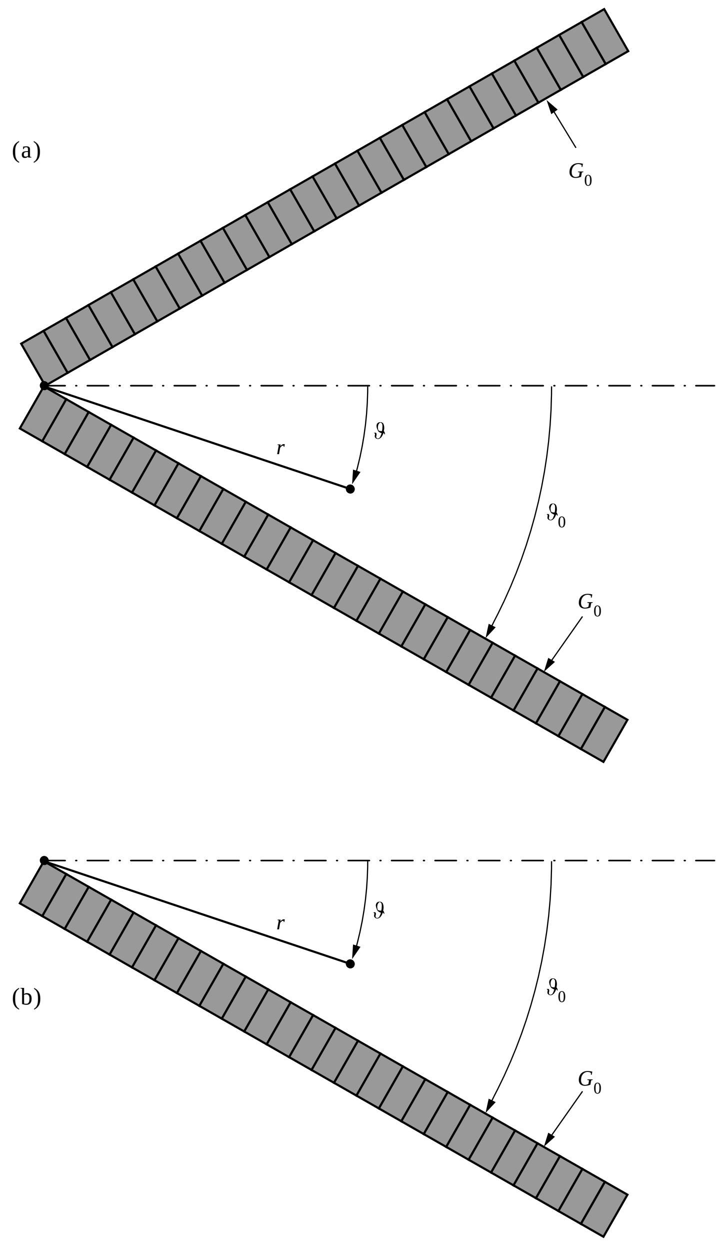


Figure 1. Wedge-shaped duct with a locally absorbing lining of normalized admittance G_0 at the boundary $\vartheta = \vartheta_0$. (a) A two-sided wedge; (b) a one-sided wedge.

or antisymmetrical with respect to the central plane. In the first case, the central plane $\vartheta = 0$ can be replaced by a rigid boundary, and in the second case by soft boundary. So one can consider the arrangement of Figure 1(b). The general case of skew fields can be treated by a superposition of symmetric and antisymmetric solutions; the arrangement with different absorbers $G_{0\pm}$ on both boundaries in Figure 1(a) can be treated by a superposition of symmetrical and antisymmetrical solutions in two arrangements with symmetrical linings; in the first subtask, the admittance of the symmetrical lining is $(G_{0+} + G_{0-})/2$ and in the second subtask it is $(G_{0+} - G_{0-})/2$. This case of different linings here will not be discussed further. Mainly symmetrical fields in a two-sided wedge with the same linings on both boundaries will be considered later. The case of antisymmetrical fields will be outlined in a section about underwater acoustic applications.

Thus, the wedge-shaped area considered has a rigid (or soft) upper boundary (at $\vartheta = 0$) and a locally reacting lower boundary at $\vartheta = \vartheta_0$ with a normalized surface admittance G_0 (normalized with respect to the characteristic wave impedance Z_0 of air). The configuration suggests using cylindrical co-ordinates r, ϑ, z; then the boundaries are co-ordinate surfaces. In many applications of the analysis derived below, the apex $r = 0$ and some region around it will not belong to the sound field; this will be used in some places. A time factor $e^{j\omega t}$ is assumed and will be dropped; $k_0 = \omega/c_0$ is the free-field wave number, $Z_0 = \rho_0 c_0$ is the characteristic impedance of air, and ρ_0 and c_0 are its density and sound speed.

Our main concern are fundamental solutions, modes, with which a modal analysis can be made in later applications. With such applications in mind we are looking for solutions which are orthogonal to each other in the angular range $(0, \vartheta_0)$.

It is it supposed that the sound field distribution in the z direction to be proportional to either $e^{\pm jk_z z}$ of $\cos(k_z z)$ of $\sin(k_z z)$ of of a linear combination of these forms with a given wave number k_z; e.g., with $k_z = 0$ for a constant field in the z direction. Then one can restrict the analysis to the two-dimensional field function $p(r, \vartheta)$. The fundamentals are the momentum equation between the particle velocity $\mathbf{v}_n$ in a direction n and the sound pressure p,

$$\mathbf{v}_n = \frac{j}{k_0 Z_0}\,\mathbf{grad}_n p; \qquad \mathbf{grad} = \left\{\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \vartheta}, \frac{\partial}{\partial z}\right\}, \tag{1}$$

the wave equation in cylindrical co-ordinates in one of the forms,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} + k^2\right)p(r, \vartheta) = 0, \qquad k^2 = k_0^2 - k_z^2,$$

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \vartheta^2} + \kappa^2\right)p(\rho, \vartheta) = 0, \qquad \rho = k_0 r, \qquad \kappa^2 = (k/k_0)^2 = 1 - (k_z/k_0)^2,$$

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\eta^2}{\rho^2}\frac{\partial^2}{\partial x^2} + \kappa^2\right)p(\rho, x) = 0, \qquad x = \eta\vartheta. \tag{2}$$

Here the variables ρ and x are introduced on the understanding that $\partial\vartheta = \eta\,\partial x$ and $\partial\rho = k_0\,\partial\rho$. Further, one has the boundary conditions at the two boundaries,

$$\mathbf{v}_\vartheta(r, 0, z) = \frac{j}{k_0 r Z_0}\frac{\partial p(r, 0, z)}{\partial \vartheta} \stackrel{!}{=} 0, \qquad G_\vartheta(\vartheta_0) \stackrel{!}{=} G_0, \tag{3}$$

together with the Sommerfeld far field condition, if the field extends to $r \to \infty$, and a special boundary condition (see below), if the apex $r = 0$ belongs to the field. We indicate by $\cdots \overset{!}{=} \cdots$ an equation which is a requirement. An expression like $z = \text{const}\,(x)$ indicates that z does not depend on x.

The quantity $G_\vartheta(\vartheta)$ in the second line or equation (3) is the azimuthal component of the normalized field admittance defined by

$$\frac{\partial p}{\partial \vartheta} = -\mathrm{j}k_0 r G_\vartheta(\vartheta) p. \tag{4}$$

For later use, the second derivative of p with respect to ϑ is written as

$$\frac{\partial^2 p}{\partial \vartheta^2} = \frac{\partial}{\partial \vartheta}\left(\frac{\partial p}{\partial \vartheta}\right) = -\mathrm{j}k_0 r\left(G_\vartheta' p + G_\vartheta \frac{\partial p}{\partial \vartheta}\right), \qquad G_\vartheta' = \frac{\partial}{\partial \vartheta} G_\vartheta,$$

$$= -(\mathrm{j}k_0 r G_\vartheta' + (k_0 r G_\vartheta)^2)p = -(\mathrm{j}\rho G_\vartheta' + (\rho G_\vartheta)^2)p. \tag{5}$$

The sound field can be separated into many tasks in cylindrical co-ordinates as $p(r, \vartheta) = T(\eta\vartheta)R(r)$ or as $p(\rho, \vartheta) = T(\eta\vartheta)R(\rho)$ with $\eta = \text{const}\,(r, \vartheta)$. Our task is to find solutions of the form $p(r, \vartheta) = T(\eta\vartheta)R(r)$, but now demanding only $\eta = \text{const}\,(\vartheta)$: i.e., permitting $\eta = \eta(r)$. Different modal solutions $p_m(r, \vartheta) = T(\eta_m\vartheta)R_m(r)$ will be orthogonal to each other in the interval $(0, \vartheta_0)$, when

$$(\partial^2/\partial\vartheta^2)T(\eta\vartheta) = -\eta^2 T(\eta\vartheta). \tag{6}$$

The boundary condition at $\vartheta = 0$ will be satisfied by

$$T(\eta\vartheta) = \left\{\begin{matrix} \cos(\eta\vartheta), & \text{symmetrical fields} \\ \sin(\eta\vartheta), & \text{antisymmetrical fields} \end{matrix}\right\},$$

$$T(x) = \left\{\begin{matrix} \cos(x), & T'(x) = -\sin(x) \\ \sin(x), & T'(x) = \cos(x) \end{matrix}\right\}, \qquad T''(x) = -T(x), \qquad x = \eta\vartheta. \tag{7}$$

Because equation (6) is sufficient to guarantee the inter-modal orthogonality in the azimuthal direction, and because equation (7) satisfies the boundary condition at the boundary $\vartheta = 0$, the relations (6) and (7) will be maintained throughout the rest of the paper.

3. CHARACTERISTIC EQUATION

The boundary condition at the absorbing boundary leads to

$$(\eta\vartheta_0)\tan(\eta\vartheta_0) = \mathrm{j}\rho\vartheta_0 G_0 = \mathrm{j}B, \qquad \text{symmetrical modes,}$$

$$(\eta\vartheta_0)\cot(\eta\vartheta_0) = -\mathrm{j}\rho\vartheta_0 G_0 = -\mathrm{j}B, \qquad \text{antisymmetrical modes.} \tag{8}$$

These are the characteristic equations for the azimuthal wave numbers η. They have the same general forms $z \tan z = \mathrm{j}B$ and $z \cot z = -\mathrm{j}B$ as the characteristic equations for the lateral wave numbers in straight lined ducts with a given quantity B. The equations have an infinite number of solutions η_m, corresponding to different modes with the mode index $m = (0), 1, 2, \ldots$ (whether the mode counting begins with $m = 0$ or $m = 1$ is to some degree arbitrary). The principal difference between equation (8) and the characteristic equations in straight lined ducts is the appearance of $\rho = k_0 r$ on the right sides. This makes η dependent on r (or ρ). Thus, $p(\rho, \vartheta) = T(\eta\vartheta)R(\rho)$ is not a complete separation, and this is the origin of the analytical problems in lined wedge-shaped ducts.

There are only a few special cases in which these problems are avoided. One of them is a dependence of the wall admittance $G_0(r) \sim 1/r$. This, however, is hardly of any practical use. The other cases are for "ideal" flanks $G_0 = 0$ (rigid) and $|G_0| = \infty$ (soft). Then the modal solutions are (for symmetrical modes) with $m = 0, 1, 2, \ldots$

$$\eta_m = m\pi/\vartheta_0 = \text{const}\,(r), \qquad \rho\vartheta_0 G_0 \to 0,$$
$$\eta_m = (m \pm 1/2)\pi/\vartheta_0 = \text{const}\,(r), \qquad |\rho\vartheta_0 G_0| \to \infty. \tag{9}$$

Such modes with $\eta_m = \text{const}\,(r)$ are called "ideal modes". It should be noticed, that ideal modes are approximated not only by $G_0 \to 0$ and $|G_0| \to \infty$, but also for $\rho \to 0$ and $\rho \to \infty$ with finite values of G_0. The wave numbers $\eta_m(r)$ are "local" in the general case. When one writes $\eta(\rho)$ and applies this in the wave equation, then the variation of ρ is understood to be produced by a variation of r and not of k_0. Thus, G_0 is constant during such variations. The right sides of equation (8) change along a straight line through the origin in the complex plane when ρ is varied.

It is important for further analysis to have available a fast-computing and safe algorithm for the solution of equation (8) for a set of η_m. "Safe" means that the algorithm should not jump betwen modes. An algorithm is described in the Appendix (see also, reference [1], chapter 26). The numerical solution makes use of the variation of the right sides of equation (8) along a straight line. For further understanding, two numerical examples are considered here. ϑ_0 is set to $\vartheta_0 = 45^\circ$ in both examples. In the first example, shown in Figure 2, the surface admittance of the boundary flank is $G_0 = 2 - 0{\cdot}5\text{j}$: i.e., it has a mass type reactance; in the second example, shown in Figure 3, the boundary admittance is $G_0 = 2 + 1\text{j}$: i.e., it has a spring type reactance. The mode order runs through $m = 0, 1, \ldots, 5$; the steps of ρ are $\Delta\rho = 0{\cdot}2$ (the points on the curves correspond to this stepping from $\rho = 0{\cdot}1$ to $\rho = 10$). The quantities η_m are plotted in both diagrams in their complex plane. All modes in the first example (mass type reactance) lean to the right with increasing ρ, beginning at $\eta_m \approx m\pi/\vartheta_0$ for small ρ values and with a tendency to approach $\eta_m = (m + 1/2)\pi/\vartheta_0$ at high ρ. In the second example (spring type reactance), however, one can distinguish three types of modes: the first mode ($m = 0$) also begins at about $\rho_m \approx m\pi/\vartheta_0$ and ends at about $\eta_m = (m + 1/2)\pi/\vartheta_0$ for increasing ρ. The second mode ($m = 1$), also beginning at $\eta_m \approx m\pi/\vartheta_0$, ends in a straight line $\eta_m \approx \rho G_0$. The higher modes ($m > 1$) lean to the left; they begin at $\eta_m \approx m\pi/\vartheta_0$ like the other modes but end at $\eta_m = (m - 1/2)\pi/\vartheta_0$. The exceptional mode $m = 1$ is a *surface wave mode*. It exists only for $\text{Im}\,\{G_0\} > 0$.

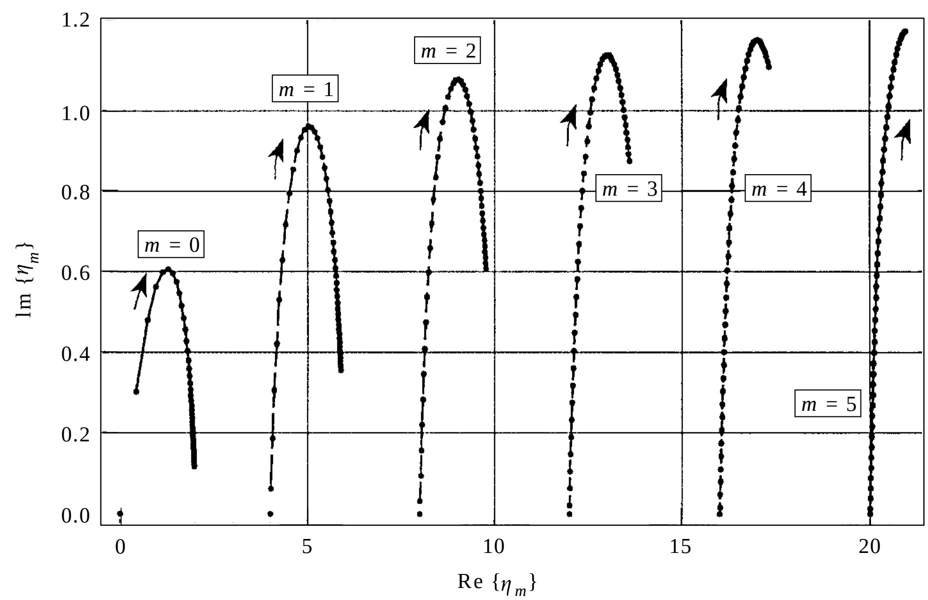


Figure 2. Solutions η_m, $m = 0, \ldots, 5$, of the characteristic equation for symmetrical modes in a wedge with a mass type reactance of the absorber. ρ is stepping through $\rho = 0{\cdot}1$–10 with $\Delta\rho = 0{\cdot}2$. $G_0 = 1 - 0{\cdot}5\mathrm{j}$, $\vartheta_0 = 45^\circ$

When one writes the characteristic equations (8) for symmetrical modes as $z \tan z = \mathrm{j}B$, this equation has branch points $z_{b,m}$ where two adjacent modes coincide, and corresponding images $B_{b,m}$ of these branch points. Values of $z_{b,m}$ and

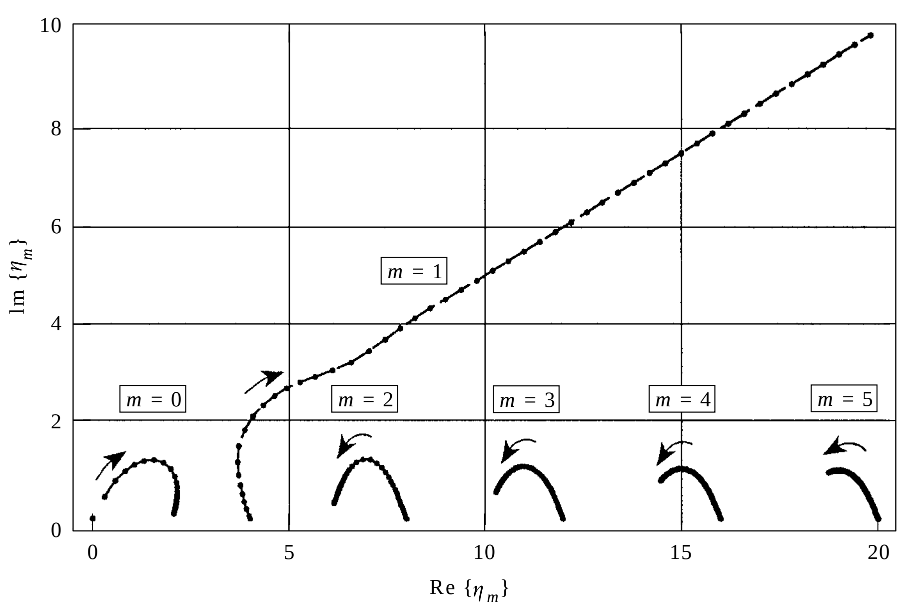


Figure 3. As Figure 2, but now with a spring type reactance of the absorber $G_0 = 2 + 1\mathrm{j}$. The second mode $m = 1$ is a surface wave mode.

$B_{b,m}$ for $m = 0, 1, \ldots, 20$ are listed in reference [1, chapter 26] and equations for curves connecting them are given there (see the Appendix). The points $B_{b,m}$ lie in the first quadrant of B and follow a curve which becomes nearly horizontal for large m. Thus, the curve of B in equations (8) with positive Im $\{G_0\}$ will pass for increasing ρ anywhere the connecting curve of the $B_{b,m}$ between $B_{b,ms}$ and $B_{b,ms+1}$. The value of m, determined by this crossing, is the order m_s of the surface wave mode.

All modes with Im $\{G_0\} \leqslant 0$ and all modes with Im $\{G_0\} > 0$ and $m < m_s$ begin at $\eta_m \approx m\pi/\vartheta_0$ and end at $\eta_m = (m + 1/2)\pi/\vartheta_0$; all modes with Im $\{G_0\} > 0$ and $m > m_s$ begin at $\eta_m \approx m\pi/\vartheta_0$ and end at $\eta_m = (m - 1/2)\pi/\vartheta_0$. One could call the first group *low modes* and the second group *high modes*. The transition of the behaviour of the surface wave mode from the shape of a low mode to the straight curve of the surface wave mode takes place at the ρ values of the mentioned crossing of the connection curve between the $B_{b,m}$. It should be noticed that the points on the curves of the low and high modes in Figures 2 and 3 lie close to each other at high ρ values: i.e., the variation of η with ρ there is small.

We resume the analysis with equation (5). With $\partial^2 p/\partial\vartheta^2 = -\eta^2 p$ one gets, from equation (5).

$$\eta^2(\rho) = \mathrm{j}\rho G'_\vartheta(\vartheta_0) + \rho^2 G_0^2. \tag{10}$$

An important finding for further analysis is the fact that $\mathrm{j}\rho G'_\vartheta$ in this equation can be approximated with good accuracy by a square polynomial in ρ:

$$\mathrm{j}\rho G'_\vartheta(\vartheta_0) = \eta^2 - (\rho G_0)^2 \approx s_0 + s_1\rho + s_2\rho^2. \tag{11}$$

Thus,

$$\eta^2 \approx s_0 + s_1\rho + (s_2 + G_0^2)\rho^2 = \varphi(\rho). \tag{12a}$$

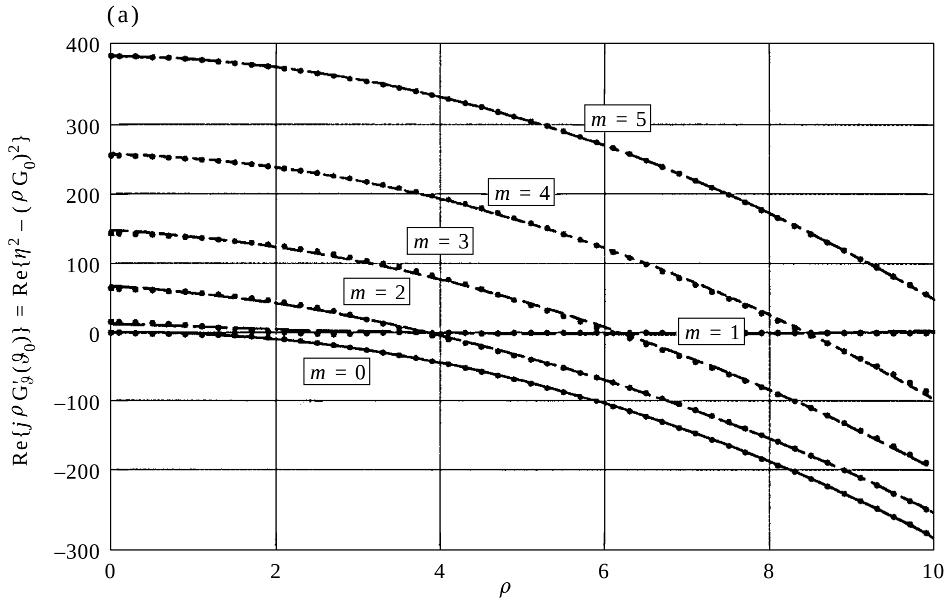
The derivatives for later use are:

$$\eta'(\rho) \approx \frac{s_1 + 2(s_2 + G_0^2)\rho}{2\eta} = \frac{\varphi'(\rho)}{2\eta},$$

$$\eta''(\rho) \approx \frac{s_2 + G_0^2 - \eta'^2}{\eta} = \frac{1}{2\eta}\left(\varphi''(\rho) - \frac{\varphi'^2(\rho)}{2\varphi(\rho)}\right). \tag{12b}$$

The coefficients s_i can be computed by regrssion (least square error fit) through the numerical values of $\eta_m(\rho)$ or by analytical approximations in the range of small ρ (see below). They depend on the mode order m. Figures 4(a, b) show the real and imaginary components of $\mathrm{j}\rho G'_\vartheta$ with the exact points from Figure 3 and the approximation curves of the polynomial (11) for $m = 0, \ldots, 5$. So one can apply from now on the square root of expression (12a) for $\eta(\rho)$.

It should be noticed that a single polynomial approximation cannot be applied over the whole range $\rho = (0, \infty)$ as will be seen immediately from the discussion of special cases, But a single polynomial can be applied over some decades of ρ.



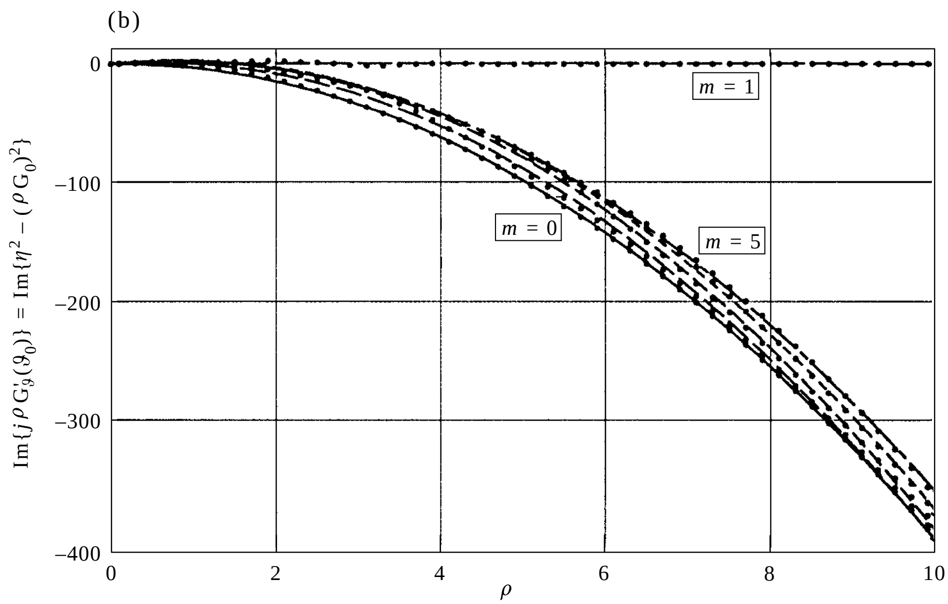


Figure 4. (a) real component and (b) imaginary component of $j\rho G_\vartheta'$ computed with the solutions of Figure 3 (points) and approximation curves with a square polynomial least square error approximation. $G_0 = 2 + 1j$, $\vartheta_0 = 45°$.

Thus, the coefficients s_i can be considered as $s_i = \text{const}\,(\rho)$ in practical tasks, and one generally only has to distinguish whether the modes shall be developed for a "low range problem" with ρ in a range $(0, \rho_{max})$ or in a "high range problem" with ρ within (ρ_{min}, ∞).

The special cases of the ideal symmetrical modes can be described by

$$G_0 \to 0: \qquad \eta_m = m\pi/\vartheta_0 = \text{const}\,(r), \qquad s_0 = \eta_m^2, \qquad s_1 = 0, \qquad s_2 = -G_0^2; \quad (13)$$

$$|G_0| \to \infty: \qquad \eta_m = (m \pm 1/2)\pi/\vartheta_0 = \text{const}\,(r), \qquad s_0 = \eta_m^2, \qquad s_1 = 0, \quad s_2 = -G_0^2. \quad (14)$$

The cases are somewhat different if G_0 is finite but $\rho \to 0$ or $\rho \to \infty$. In the case $\rho \to 0$, one inserts the polynomial representation for $\eta(\rho)$ into the left sides of equations (8), makes a power series development of that side and compares coefficients of corresponding powers of ρ (the right sides of equations (8) are interpreted as power series in ρ). The result is

$$G_0 \neq 0, \qquad \rho \to 0: \qquad s_0 = (m\pi/\vartheta_0)^2, \qquad s_1 = \delta_m \mathrm{j} G_0/\vartheta_0, \qquad \delta_m = \begin{Bmatrix} 1, & m = 0 \\ 2, & m > 0 \end{Bmatrix},$$

$$s_2 = \begin{Bmatrix} -G_0^2(1 - 1/(m\pi)^2), & m \neq 0 \\ -G_0^2 5/8, & m = 0 \end{Bmatrix}. \quad (15)$$

This is the above-mentioned analytical approximation for the coefficients s_i at small ρ. The case $\rho \to \infty$, if the mode is not the surface wave mode, is described by

$$|\rho G_0| \to \infty: \eta_m = (m \pm 1/2)\pi/\vartheta_0, \qquad s_0 = \eta_m^2, \qquad s_1 = 0, \qquad s = -G_0^2, \quad (16)$$

and the case $\rho \to \infty$ with the surface wave mode

$$\eta_m \approx \rho G_0, \qquad \mathrm{j}\rho G'_\vartheta(\vartheta_0) \approx 0, \qquad s_0 = s_1 = s_2 = 0. \quad (17)$$

These expressions can be taken as analytical approximations to the s_i in the range of large ρ. So the s_i take different values depending on $\rho \to 0$ or $\rho \to \infty$; thus, they are not strictly $s_i = \text{const}\,(\rho)$, but with a sufficient precision over a restricted range of ρ.

This section ends with the integral of orthogonality of modes:

$$\frac{1}{\vartheta_0} \int_0^{\vartheta_0} T_m(r, \vartheta) T_n(r, \vartheta)\, \mathrm{d}\vartheta = \delta_{m,n} N_m,$$

$$N_m = \frac{1}{2}\left[1 \pm \frac{\sin(2\eta_m \vartheta_0)}{2\eta_m \vartheta_0}\right], \qquad \begin{Bmatrix} \text{symmetrical} \\ \text{antisymmetrical} \end{Bmatrix}, \quad (18)$$

with the special value $N_0 = 1$ when the mode $m = 0$ is a symmetrical ideal mode.

4. WAVE EQUATION

Inserting the form $p(\rho,x) = T(x)R(\rho)$, $x = \eta\vartheta$, into wave equation (2) gives

$$T(x)\left[R(\rho)\left(\kappa^2 - \frac{\eta^2}{\rho^2}\right) + \frac{1}{\rho}R'(\rho) + R''(\rho)\right]$$

$$+ \vartheta T'(x)\left[R(\rho)\left\{\frac{\eta'}{\rho} + \eta''\right\} + 2\eta' R'(\rho)\right] - T(x)R(\rho)(\vartheta\eta')^2 = 0, \quad (19)$$

where the primes indicate derivatives with respect to the argument, which is ρ for $\eta(\rho)$. The first line is called *main part* and the second line the *rest* of the wave equation. It can be seen that the rest vanishes when $\eta = \text{const}\,(\rho)$, i.e., for ideal modes and also in the limit $\rho \to \infty$ for $G_0 \neq 0$, because then $\eta \approx \text{const}\,(\rho)$ (if it is not a surface wave mode). Further, the rest vanishes for $\vartheta \to 0$; both terms with the order ϑ^2 for symmetrical modes, and the first term with the order ϑ, the second term with the order ϑ^3 for antisymmetrical modes.

Instead of making zero the sum of the main part and of the rest, now assume that both parts shall be zero (this is a sufficient condition for the solution of the wave equation, but not a necessary one). The main part gives the radial differential equation

$$R(\rho)\left(\kappa^2 - \frac{\eta^2}{\rho^2}\right) + \frac{1}{\rho}R'(\rho) + R''(\rho) = 0. \quad (20)$$

Field components with the form $p(\rho, \vartheta) = T(\eta\vartheta)R(\rho)$ in which $R(\rho)$ satisfies equation (20) and $\eta(\rho)$ are solutions of the characteristic equation are called *fictitious modes*, and the solutions of the full wave equation and boundary condition equations are called *true modes* or *wedge modes*. Ideal modes thus are subspecies of fictitious modes, and they are *true modes* or *wedge modes* under ideal boundary conditions, because they satisfy the wave equation and the ideal boundary conditions. The other modes are fictitious, because they produce a non-zero rest of the wave equation. All fictitious modes become true modes near the plane $\vartheta = 0$, because there they approximately satisfy the full wave equation and the boundary conditions. The differential equation (20) can be solved with the form (12a) for $\eta(\rho)$ (see the next section).

Under the condition of very small ϑ_0 of underwater acoustics, mentioned in the introduction, the fictitious modes (then antisymmetrical) would be suitable elementary solutions for the synthesis of sound fields with no restrictions concerning $k_0 r$, when the seabed can be described by a locally reacting absorber. This application will be discussed in a later section.

Under the more general conditions of duct acoustics, we have to make zero the second line of equation (19), the rest of the wave equation. This is done by two methods. In the first method, we replace $R(\rho) \to R(\rho) + r(\rho)$ for the radial function of a mode and compose the "rest" $r(\rho)$ by radial functions of other fictitious modes. In the second method, $r(\rho)$ is derived as a new function from the rest of the wave equation after having made some assumptions of approximation.

5. FICTITIOUS MODES

The approximation (12a) for η^2 has the advantage that the differential equation (20) can be solved. For $\eta = \text{const}\,(\rho)$ equation (20) is the Bessel differential equation. So one expects for $R(\rho)$ of the ideal modes cylinder functions $Z_\eta(\kappa\rho) = Z_\eta(kr)$ of the order η. For $\eta \neq \text{const}\,(\rho)$ equation (20) assumes the form

$$\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho} + (\gamma^2 - (s_0/\rho^2 + s_1/\rho))R(\rho) = 0, \qquad \gamma^2 = \kappa^2 - (s_2 + G_0^2) = \text{const}\,(\rho). \tag{21}$$

A fundamental system of solutions are the functions.

$$\mathrm{R}_{1,2}(\rho) = \rho^{\sqrt{s_0}}\,\mathrm{e}^{-\mathrm{j}\rho\gamma}\left\{\begin{matrix} \mathrm{U}\left(\frac{1}{2} + \sqrt{s_0} - \frac{\mathrm{j}s_1}{2\gamma}\,,\,1 + 2\sqrt{s_0},\,2\mathrm{j}\rho\gamma\right) \\ M\left(\frac{1}{2} + \sqrt{s_0} - \frac{\mathrm{j}s_1}{2\gamma}\,,\,1 + 2\sqrt{s_0},\,2\mathrm{j}\rho\gamma\right) \end{matrix}\right\}, \tag{22}$$

where $\mathrm{U}(a, b, z)$ and $\mathrm{M}(a, b, z)$ are *confluent hypergeometric functions*; $\mathrm{M}(a, b, z)$ are also called *Kummer's function*, and $\mathrm{U}(a, b, z)$ is called the *Tricomi function*. Besides the symbol $\mathrm{U}(a, b, z)$, $\Psi(a, b; z)$ is also used in the literature, and for $\mathrm{M}(a, b, z)$ also $\Phi(a, b; z)$ or ${}_1\mathrm{F}_1(a, b; z)$. $\mathrm{U}(a, b, z)$ exists everywhere, especially also for $|z|\to\infty$; it is multi-valued; its principal branch is made definite by a branch cut along the negative real axis of the z-plane; its behaviour at $z\to 0$ dramatically changes with the parameters (like Hankel functions). The function $\mathrm{M}(a, b, z)$ is simpler for $z\to 0$, but it increases exponentially for complex z with $|z|\to\infty$ (like Bessel functions). We are searching for a pair of fundamental solutions of $R(x)$ which for $G_0\to 0$ will reproduce as solutions the Hankel functions $H^{(1)}_{m\pi/\vartheta_0}(\rho)$, $H^{(2)}_{m\pi/\vartheta_0}(\rho)$ (up to possible constant factors, unimportant in this context) which are known to be the solutions in that special case. Such a pair of fundamental solutions is

$$\mathrm{R}^{(1)}(\rho) = \rho^{\sqrt{s_0}}\,\mathrm{e}^{+\mathrm{j}\rho\gamma}\mathrm{U}\left(\frac{1}{2} + \sqrt{\mathrm{s}_0} + \frac{\mathrm{j}s_1}{2\gamma}\,,\,1 + 2\sqrt{s_0},\,-2\mathrm{j}\rho\gamma\right),$$

$$R^{(2)}(\rho) = \rho^{\sqrt{s_0}}\,\mathrm{e}^{-\mathrm{j}\rho\gamma}\mathrm{U}\left(\frac{1}{2} + \sqrt{\mathrm{s}_0} - \frac{\mathrm{j}s_1}{2\gamma}\,,\,1 + 2\sqrt{s_0},\,+2\mathrm{j}\rho\gamma\right), \tag{23a}$$

in which only the root γ changes its sign or—equivalently—the sign of the explicitly appearing $\pm j$ is changed (as in Hankel functions). The solution $\mathrm{R}^{(1)}(\rho)$ produces an inward propagating wave (towards the apex, like Hankel functions of the first kind with our convention of the time factor); the solution $R^{(2)}(\rho)$ produces an outward propagating wave (like Hankel functions of the second kind). They satisfy Sommerfeld's far field condition, if one computes the root so that

$$\gamma := \sqrt{\kappa^2 - (s_2 + G_0^2)}, \qquad \mathrm{Im}\,\{\gamma\} < 0, \qquad \mathrm{Re}\,\{\gamma\} > 0 \tag{24}$$

holds, where the first prescription has priority and the second prescription is applied when the root is real. A different form of writing is

$$R^{(1)}(\rho) = A(\rho)\mathrm{U}(a, b, z), \qquad R^{(2)}(\rho) = A(\rho)(\mathrm{e}^{z}\mathrm{U}(b-a, b, -z)), \tag{23b}$$

$$A(\rho) = \rho^{\sqrt{s_0}}\,\mathrm{e}^{+\mathrm{j}\rho\gamma} = \rho^{\sqrt{s_0}}\,\mathrm{e}^{-z/2},$$

$$a = \tfrac{1}{2} + \sqrt{s_0} + \mathrm{j}s_1/2\gamma, \qquad b = 1 + 2\sqrt{s_0}, \quad z = -2\mathrm{j}\rho\gamma. \tag{23c}$$

Because the coefficients s_n will depend on the mode order m, a more complete notation for $R^{(i)}$ is $R_m^{(i)}(\rho)$, $i = 1, 2$. These solutions change over to the Hankel functions $H_\nu^{(i)}(z)$, $i = 1, 2$ (up to the constant factors) for $s_1 \to 0$, because of the relation

$$\mathrm{H}_\nu^{(1,2)}(y) = \mp(2\mathrm{j}/\sqrt{\pi})(2y)^\nu\,\mathrm{e}^{\pm\mathrm{j}(y-\nu\pi)}\mathrm{U}(1/2+\nu, 1+2\nu, \mp 2\mathrm{j}y), \tag{25}$$

with which we obtain for our fundamental solutions in the case $s_1 = 0$

$$R^{(1,2)}(\rho) \xrightarrow[s_1 \to 0]{} \pm\,\mathrm{j}\,\frac{\sqrt{\pi}}{2}\,\frac{\mathrm{e}^{\pm\mathrm{j}\pi\sqrt{s_0}}}{(2\gamma)^{\sqrt{s_0}}}\,\mathrm{H}^{(1,2)}_{\sqrt{s_0}}(\rho\gamma). \tag{26}$$

The fundamental solutions $R_m^{(i)}(\rho)$ of equation (23) which are similar to Hankel functions can be used when the apex of the wedge does not belong to the field range. If it is included, the form of equation (22) with $\mathrm{M}(a, b, z)$ is preferable. The derivative $R_m'^{(1)}(\rho)$ for $i = 1, 2$ can be written as

$$R'^{(i)}(\rho) = \left(\frac{\sqrt{s_0}}{\rho} \pm \mathrm{j}\gamma\right)R_m^{(1,2)}(\rho) \mp 2\mathrm{j}\gamma\rho^{\sqrt{s_0}}\,\mathrm{e}^{\pm\mathrm{j}\rho\gamma}\begin{Bmatrix}\mathrm{U}'(a, b, z)\\ \mathrm{U}'(b-a, b, -z)\end{Bmatrix}, \tag{27}$$

in which the derivatives $\mathrm{U}'(a, b, z) = \mathrm{dU}(a, b, z)/\mathrm{d}z$ can be computed by one of the relations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{U}(a, b, z) &= \mathrm{U}(a, b, z) - \mathrm{U}(a, b+1, z)\\ &= \frac{1}{z}[(1-b)\mathrm{U}(a, b, z) - (1+a-b)\mathrm{U}(a, b-1, z)]\\ &= \frac{1}{z}[(1-b+z)\mathrm{U}(a, b, z) - \mathrm{U}(a-1, b, z)]\\ &= \frac{1}{z}[(1-b+z)\mathrm{U}(a, b, z) - \mathrm{U}(a-1, b-1, z)].\end{aligned} \tag{28}$$

Some special cases shall be considered. The first special case is the rigid boundary $G_0 = 0$. One gets, from equations (13) and (26),

$$\mathrm{R}_m^{(i)}(k_0 r) = \pm \frac{\mathrm{j}\sqrt{\pi}}{2} \frac{\mathrm{e}^{\pm \mathrm{j}\pi^2 m/\vartheta_0}}{(2\kappa)^{m\pi/\vartheta_0}} \mathrm{H}^{(i)}_{m\pi/\vartheta_0}(kr), \qquad i = 1, 2, \tag{29a}$$

which—up to constant factors—are indeed the Hankel functions of fractional orders. A different special case is encountered when G_0 remains finite but $\rho = k_0 r \to 0$. Then—as can be seen from equation (15)—the coefficient s_1 remains finite. An important special case arises, when $|k_0 r G_0| \to \infty$, i.e., either when $|G_0| \to \infty$, or, more important for us, when $k_0 r \to \infty$ with a finite value of G_0. Then with equation (16), the solutions for modes which are not the surface wave mode (see below for that case) become

$$\mathrm{R}_m^{(i)}(k_0 r) = \pm \mathrm{j} \frac{\sqrt{\pi}}{2} \frac{\mathrm{e}^{\pm \mathrm{j}\pi\eta_m}}{(2\kappa)^{\eta_m}} \mathrm{H}^{(i)}_{\eta_m}(kr), \qquad i = 1, 2. \tag{29b}$$

They are of the same form as equation (29a) but now with different values of η_m. In the special case of a surface wave mode with high $k_0 r$ values, i.e., for $\eta_m \approx k_0 r G_0$, one gets with equation (17)

$$\mathrm{R}^{(i)}(k_0 r) = \pm \mathrm{j} \frac{\sqrt{\pi}}{2} \mathrm{H}_0^{(i)}(k_0 r \sqrt{\kappa^2 - G_0^2}), \qquad i = 1, 2. \tag{29c}$$

The same result is obtained if $G_\vartheta(\vartheta) = \mathrm{const}\,(\vartheta) = G_0$ is supposed for all ϑ including $\vartheta = \vartheta_0$, and also for the weaker requirement $G'_\vartheta(\vartheta_0) = 0$.

6. THE REST $r(\rho)$ AS A SUPERPOSITION OF FICTITIOUS MODES

Assume a mode to be of the form

$$p_m^{(i)}(\rho, \vartheta) = T(\eta_m \vartheta) \mathrm{R}_m^{(i)}(\rho) + \sum_n T(\eta_n \vartheta)(c_n^{(1)} \mathrm{R}_n^{(1)}(\rho) + c_n^{(2)} \mathrm{R}_n^{(2)}(\rho)),$$

$$i = 1, 2, \qquad c_m^{(i)} = 0, \tag{30a}$$

which also can be written as

$$p_m^{(i)}(\rho, \vartheta) = \sum_n T(\eta_n \vartheta)(c_n^{(1)} \mathrm{R}_n^{(1)}(\rho) + c_n^{(2)} \mathrm{R}_n^{(2)}(\rho)), \qquad i = 1, 2, \qquad c_m^{(i)} = 1. \tag{30b}$$

Thus, the mth inward ($i = 1$) or outward ($i = 2$) propagating mode in the wedge is composed of the mth fictitious mode and a rest, which itself is composed of other fictitious modes. Inserting this in equation (19) makes the first line zero and gives for the rest of the wave equation (the second line of equation (19)), after division by ϑ^2,

$$\sum_n T(\eta_n\vartheta)\eta_n'^2(c_n^{(1)}\mathrm{R}_n^{(1)}(\rho)+c_n^{(2)}\mathrm{R}_n^{(2)}(\rho))$$

$$\overset{!}{=} \sum_n \frac{T'(\eta_n\vartheta)}{\vartheta}\left[\left(\frac{\eta_n'}{\rho}+\eta_n''\right)(c_n^{(1)}\mathrm{R}_n^{(1)}(\rho)+c_n^{(2)}\mathrm{R}_n^{(2)}(\rho)) + 2\eta_n'(c_n^{(1)}\mathrm{R}_n'^{(1)}(\rho)+c_n^{(2)}\mathrm{R}_n'^{(2)}(\rho))\right]. \tag{31}$$

The task is to determine the coefficients $c_n^{(k)}$. This can be done only approximately. As a first possible approximation, one would suppose equation (31) to hold only at the boundary ϑ_0 and rely on the fact that the rest will vanish automatically near $\vartheta = 0$ (see above). This assumption would deliver one equation continuous in ρ. It could be the basis of a collocation method: equation (31) is written at enough distinct values ρ to give enough linear equations for the computation of $c_n^{(k)}$. We prefer a second method in which the orthogonality of the $T(\eta_n\vartheta)$ is applied; it minimizes the square error in the ϑ space. Multiplying both sides of equation (31) by $T(\eta_\mu\vartheta)$, $\mu = 0, 1, \ldots$, and integrating over the interval $(0, \vartheta_0)$ gives the linear inhomogeneous system of equations

$$N_\mu\eta_\mu'^2(c_\mu^{(1)}\mathrm{R}_\mu^{(1)}(\rho)+c_\mu^{(2)}\mathrm{R}_\mu^{(2)}(\rho)) = \sum_n S_{\mu n}\left[\left(\frac{\eta_n'}{\rho}+\eta''_n\right)(c_n^{(1)}\mathrm{R}_n^{(1)}(\rho)+c_n^{(2)}\mathrm{R}_n^{(2)}(\rho)) + 2\eta_n'(c_n^{(1)}\mathrm{R}_n'^{(1)}(\rho)+c_n^{(2)}\mathrm{R}_n'^{(2)}(\rho))\right], \tag{32}$$

with the norms N_μ of the modes from equation (18) and the integrals

$$S_{m,n} = \frac{1}{\vartheta_0}\int_0^{\vartheta_0} T(\eta_m\vartheta)\,\frac{T'(\eta_n\vartheta)}{\vartheta}\,\mathrm{d}\vartheta, \tag{33a}$$

which in the case of symmetrical modes evaluate to

$$S_{m,n} = (1/2\vartheta_0)(\mathrm{Si}\,((\eta_m-\eta_n)\vartheta_0) - \mathrm{Si}\,((\eta_m+\eta_n)\vartheta_0)), \tag{33b}$$

with the sine integral function $\mathrm{Si}\,(z)$. Equation (32) is a linear inhomogeneous system of equations for $c_n^{(k)} \neq c_m^{(i)}$; they are still continuous in ρ. Orthogonality integrals over $\mathrm{R}_n^{(k)}(\rho)$, $k = 1, 2$, are not known. We would like to weight the similarity of the radial functions in equation (32) with the radial functions $\mathrm{R}_m^{(k)}(\rho)$ of the wanted mode m. Therefore, both sides of equation (32) are multiplied by $\mathrm{R}_m^{(k)}(\rho)$, $k = 1, 2$, and integrated over the interval of ρ for which the coefficients s_i were determined. This gives two linear inhomogeneous systems of equations (now with coefficients = const (ρ)) for the two sets of coefficients $c_n^{(k)} \neq c_m^{(i)}$. The integrals must be evaluated numerically, because analytic integrals over products of the $R_n^{(k)}(\rho)$ were not found.

It is this need of numerical integrations which makes tedious the described first method of evaluation of the modal rest $r(\rho)$. On the other hand, it has some analytical advantages over the second method, to be described in the next section. It does not make additional assumptions concerning the magnitude of the rest $r(\rho)$; it minimizes the rest of the wave equation in the sense of minimal square errors in the wedge area; it gives explicitly the contribution of the reflections (by the contributions of the terms with $R_n^{(k)}(\rho)$ with $k \neq i$) in the mode m, i.

7. SEPARATE SOLUTION OF THE REST OF THE WAVE EQUATION

To compensate for the rest of the wave equation for larger values of ϑ_0 (second line in equation (19)), the wanted solution of the sound field is formulated as

$$p(\rho, \vartheta) = T(\eta\vartheta)(R(\rho) + \vartheta^{2n} r(\rho)), \tag{34}$$

in which $R(\rho)$ is a solution of equation (20) and $r(\rho)$ must still be determined. With $n = 0$, this would produce in equation (19) similar terms on the left side with the replacement $R(\rho) \rightarrow r(\rho)$. But now the first line would not vanish, because $r(\rho)$ must not be a solution of equation (20). The rest equations of both $R(\rho)$ and $r(\rho)$ would vanish for $\vartheta \rightarrow 0$. We suppose, as first approximation, that it will be sufficient if the rests of equation (19) should be zero for $\vartheta = \vartheta_0$ (and not in the full wedge space). Further, numerical results for $r(\rho)$ are anticipated and it is assumed that it varies slowly with ρ so that $r''(\rho)$ can be neglected. Even then the first line of equation (19) for $r(\rho)$ would give a contribution at $\vartheta \rightarrow 0$ if one takes $n \leqslant 1$ in equation (34). Therefore, the power of ϑ in equation (34) must be positive, and because $p(\rho, \vartheta)$ must be even in ϑ (for symmetrical modes), the lowest possible value is $n = 2$ (then indeed $\vartheta^4 r''(\rho)$ is neglected); the value $n = 2$ is taken in what follows.

The rest equation with these approximations assumes the form

$$r(\rho)g(\rho) + r'(\rho)h(\rho) = f(\rho), \tag{35}$$

with the known functions

$$g(\rho) = \vartheta_0^2\left[\frac{12}{\rho^2} + \vartheta_0^2\left(\kappa^2 - \frac{\eta^2}{\rho^2}\right) - \vartheta_0^4\eta'^2\right] - \mathrm{j}G_0\vartheta_0^3\left[\frac{8}{\rho} + \vartheta_0^2\,\frac{\eta' + \rho\eta''}{\eta}\right], \tag{36}$$

$$h(\rho) = \vartheta_0^4\left(\frac{1}{\rho} - 2\mathrm{j}\vartheta_0 G_0\,\frac{\rho\eta'}{\eta}\right), \tag{37}$$

$$f(\rho) = R(\rho)\mathrm{j}\vartheta_0\left[G_0\,\frac{\eta' + \rho\eta''}{\eta} - \mathrm{j}\vartheta_0\eta'^2\right] + R'(\rho)2\mathrm{j}\vartheta_0 G_0\,\frac{\rho\eta'}{\eta}\,. \tag{38}$$

A general solution of equation (35) is

$$r(\rho) = \mathrm{e}^{-I(\rho)}\left[c + \int^{\rho} \mathrm{e}^{I(x)} \frac{f(x)}{h(x)}\,\mathrm{d}x\right] = \mathrm{C}\,\mathrm{e}^{-I(\rho)} + \int^{\rho} \mathrm{e}^{-\int_x^{\rho} g(t)/h(t)\,\mathrm{d}t} \frac{f(x)}{h(x)}\,\mathrm{d}x, \quad (39a)$$

with

$$I(x) = \int \frac{g(x)}{h(x)}\,\mathrm{d}x. \quad (39b)$$

The integrals can be assigned suitable lower integration limits (e.g., the lower limit $\rho_{\min}$ of the task range); this will influence only the integration constant C. The integration constant C shall be determined from boundary conditions in ρ (depending on whether $\rho = 0$ or $\rho = \infty$ belong to the field area), or it can be used to support the assumption of small $r''(\rho)$ (see below). The integral $I(x)$ can be evaluated analytically; its value is

$$I(x) = ax + b \ln x - (1/4) \ln (s_0 + s_1 x + (s_2 + G_0^2)x^2) + \sum_{n=1}^{3} c_n \ln (x - x_n), \quad (40a)$$

with the abbreviations

$$a = (\mathrm{j}/2\vartheta_0 G_0)(\kappa^2 - (s_2 + G_0^2)(1 + \vartheta_0^2)), \qquad b = 12/\vartheta_0^2 - s_0,$$

$$c_n = \frac{1}{4\vartheta_0 G_0} \frac{\alpha + \beta x_n + \gamma x_n^2}{\mathrm{j}s_1 + 2(\mathrm{j}(s_2 + G_0^2) + \vartheta_0 G_0 s_1)x_n + 6\vartheta_0 G_0 (s_2 + G_0^2)x_n^2}, \quad (40b)$$

$$\alpha = \mathrm{j}\vartheta_0 G_0 s_1 + s_0(32G_0^2 + 2\kappa^2 - 4\mathrm{j}\vartheta_0 G_0 s_1 - 2(s_2 + G_0^2)(1 + \vartheta_0^2)),$$

$$\beta = 2\mathrm{j}\vartheta_0 G_0 (s_2 + G_0^2) + 4\vartheta_0 G_0 s_0 (\mathrm{j}(\kappa^2 - (s_2 + G_0^2)) + \vartheta_0 G_0 s_1) - \mathrm{j}\vartheta_0 G_0 s_1^2 (2 - \mathrm{j}\vartheta_0)^2$$
$$+ s_1(2\kappa^2 - 16G_0^2 + 2\vartheta_0^2 G_0^2 - 2(s_2 + G_0^2)(1 + \vartheta_0^2)),$$

$$\gamma = 8\vartheta_0^2 G_0^2 s_0 (s_2 + G_0^2) - 2(s_2 + G_0^2)^2 (1 + \vartheta_0^2) + (s_2 + G_0^2)(2\kappa^2 - 64G_0^2 + 8\vartheta_0^2 G_0^2)$$
$$+ 2\mathrm{j}\vartheta_0 G_0 s_1 (\kappa^2 - (s_2 + G_0^2)(3 + \vartheta_0)), \quad (40c)$$

and the x_n are the solutions of

$$s_0 + s_1 x + (s_2 + G_0^2 + \vartheta_0 G_0 s_1)x^2 - 2\mathrm{j}\vartheta_0 G_0 (s_2 + G_0^2)x^3 = 0. \quad (40d)$$

They can either be evaluated numerically or written in a closed (however lengthy) form. The exponential factor in $r(\rho)$ thus becomes

$$\mathrm{e}^{+I(x)} = \frac{\mathrm{e}^{ax} x^b}{(s_0 + s_1 x + (s_2 + G_0^2)x^2)^{1/4}} \prod_{n=1}^{3} (x - x_n)^{c_n} = \frac{\mathrm{e}^{a,x} x^b}{\sqrt{\eta(x)}} \prod_{n=1}^{3} (x - x_n)^{c_n}. \quad (41)$$

The limit value at $x \to 0$ is

$$\mathrm{e}^{+I(x)} \xrightarrow[x\to 0]{} \frac{x^b}{s_0^{1/4}}(-x_1)^{c_1}(-x_2)^{c_2}(-x_3)^{c_3}. \tag{42}$$

The real component of b should be non-negative to make this regular.

The constant of integration C in $r(\rho)$ must still be determined. If one is constructing modes for a high range application, i.e., where $\rho \to \infty$ shall be included in the range of the modes, it is a question of whether a suitable value of C is needed to satisfy the requirement $r(\rho) \to 0$ for $\rho \to \infty$. For that, consider the behaviour of the components of $r(\rho)$ for $\rho \to \infty$ and make use of equations (16) and (17). One finds for $\rho \to \infty$ of a mode which is not a surface wave mode

$$g(x) \xrightarrow[x\to\infty]{} \vartheta_0^2\left[\kappa^2\vartheta_0^2 - \frac{8\mathrm{j}G_0\vartheta_0}{x} + \frac{12-\eta_m^2\vartheta_0^2}{x^2}\right] \to \kappa^2\vartheta_0^4, \qquad h(x) \xrightarrow[x\to\infty]{} \frac{\vartheta_0^4}{x} \to 0, \tag{43a, b}$$

$$I(x) \xrightarrow[x\to\infty]{} -\frac{8\mathrm{j}G_0}{\vartheta_0}x + \frac{\kappa^2}{2}x^2 + (12/\vartheta_0^2 - \eta_m^2)\ln x, \tag{43c}$$

$$\frac{f(x)}{h(x)} \xrightarrow[x\to\infty]{} \frac{\mathrm{j}(s_2+G_0^2)}{G_0\vartheta_0^3}R(x) - \frac{1}{\vartheta_0^4}R'(x) \to -\frac{1}{\vartheta_0^4}R'(x). \tag{43d}$$

As $\kappa^2/2$ is positive, $\exp(-I(x))$ will vanish for $x \to \infty$ super-exponentially and $r(\rho) \to 0$ will be satisfied wthout a special choice of C. Thus, one can apply the freedom in C for a minimization of $r''(\rho)$. This second derivative is given by

$$r''(\rho) = \frac{1}{h}\left[f' - \mathrm{e}^{-I(\rho)}\left(c + \int^{\rho}\mathrm{e}^{+I(x)}\frac{f}{h}\,\mathrm{d}x\right)\left(g' - \frac{g}{h}(g+h')\right) - \frac{f}{h}(g+h')\right]. \tag{44}$$

Because C shall be constant in ρ, one can demand the average square to be zero,

$$\langle |r''(\rho)|^2\rangle \overset{!}{=} 0, \tag{45}$$

and compute C from that.

In the case of $\rho \to \infty$ with the surface wave mode one finds

$$g(x) \xrightarrow[x\to\infty]{} \vartheta_0^4(\kappa^2 - G_0^2(1+\vartheta_0^2)) - \mathrm{j}G_0\vartheta_0^3(8+\vartheta_0^2)\frac{1}{x} + \frac{12\vartheta_0^2}{x^2} \to \vartheta_0^4(\kappa^2 - G_0^2(1+\vartheta_0^2)), \tag{46a}$$

$$h(x) \xrightarrow[x\to\infty]{} \vartheta_0^4(-2\mathrm{j}G_0\vartheta_0 + 1/x) \to -2\mathrm{j}G_0\vartheta_0^5, \qquad \frac{g(x)}{h(x)} \xrightarrow[x\to\infty]{} \mathrm{j}\,\frac{\kappa^2 - G_0^2(1+\vartheta_0^2)}{2G_0\vartheta_0}, \tag{46b, c}$$

$$I(x) \xrightarrow[x\to\infty]{} \frac{\kappa^2 - G_0^2(33-\vartheta_0^2)}{8G_0^2\vartheta_0^2}(\mathrm{j}\pi + \ln(4G_0^2\vartheta_0^2)) + \mathrm{j}\frac{\kappa^2 - G_0^2(1+\vartheta_0^2)}{2G_0\vartheta_0}x$$

$$+\left(\frac{12}{\vartheta_0^2} + \frac{\kappa^2 - G_0^2(33-\vartheta_0^2)}{4G_0^2\vartheta_0^2}\right)\ln x, \quad (46\mathrm{d})$$

$$\mathrm{e}^{I(x)} \xrightarrow[x\to\infty]{} \mathrm{e}^{\mathrm{j}(\gamma^2 - G_0^2\vartheta_0^2)/(2G_0\vartheta_0)x}, \qquad \frac{f(x)}{h(x)} \xrightarrow[x\to\infty]{} \frac{\mathrm{j}G_0}{\vartheta_0^3}R(x) - \frac{1}{\vartheta_0^4}R'(x). \quad (46\mathrm{e, f})$$

Thus, with $R(x)$ from equation (29c), the sign convention (24) for γ and the sign of $\mathrm{Im}\,\{G_0\} > 0$ in the case of the surface wave mode, it follows that $r(\rho)\to 0$ for $\rho\to\infty$. This could have been expected in advance, because the surface wave mode clings to the absorbing boundary and does not "see" the other boundary if that is distant enough.

If one intends to construct modes for which the origin $\rho = 0$ belongs to the field area, then C is determined from the conservation of the volume flow near the apex. The volume flow $q_r = v_r r\vartheta_0 = r\vartheta_0 \partial p/\partial r$ must equal the volume flow $q_0 = rv_\vartheta = rG_0 p$ into the absorber for $r\to 0$. The radial function $R_m(\rho)$ should be taken from equation (22) with the function $\mathrm{M}(a, b, z)$.

The construction of wedge modes with the help of the rest function $r(\rho)$ described in this section is less intensive in numerical computations than the construction of wedge modes in the previous section. It further has the "appeal" that it corrects to some degree numerical errors in the coefficients s_i.

8. UNDERWATER ACOUSTIC CONDITIONS

Under this heading the conditions of a soft boundary at $\vartheta = 0$, small wedge angles ϑ_0 and large $\rho = k_0 r$ values are briefly described. It shall not be discussed here whether a model with a locally reacting seabed is a good model. The azimuthal shape of the modes is $T(\eta_m\vartheta) = \sin(\eta_m\vartheta)$; the characteristic equation is

$$(\eta\vartheta_0)\cot(\eta\vartheta_0) = -\mathrm{j}\rho\vartheta_0 G_0. \quad (47)$$

The approximation (12a) for η^2 can also be applied here, and surface wave modes can exist here too. A mode formulation $p_m(\rho, \vartheta) = T(\eta_m\vartheta)R_m(\rho)$ when inserted in the wave equation gives again equation (19). The last term in the second line of that equation now disappears for $\vartheta\to 0$ as ϑ^3 and the first term in the second line as ϑ. If one is content with a first order approximation, then the fictitious modes with $R_m(\rho)$ taken from equation (22) or equation (23) are wedge modes in that approximation. In a third order of approximation for small wedge angles ϑ_0, it is sufficient to null the first term of the second line. Upon assuming a composition of the wedge mode as in equation (30), equation (31) is replaced by the simpler equation

$$\sum_n T'(\eta_n\vartheta)\left[\left(\frac{\eta_n'}{\rho}+\eta_n''\right)(c_n^{(1)}\mathrm{R}_n^{(1)}(\rho)+c_n^{(2)}\mathrm{R}_n^{(2)}(\rho))+2\eta_n'(c_n^{(1)}\mathrm{R}_n'^{(1)}(\rho)+c_n^{(2)}\mathrm{R}_n'^{(2)}(\rho))\right]\stackrel{!}{=}0. \quad (48)$$

The terms of the sum are orthogonal over $(0, \vartheta_0)$, so the brackets should vanish individually. One can construct a linear inhomogeneous system of equations for the coefficients $c_n^{(k)}$ as described above. It would be interesting to apply only the mode $T(\eta_m\vartheta)\mathrm{R}_m^{(k\neq i)}(\rho)$ for correction: i.e., the mode with the same index m but with the opposite direction of propagation. The equation for the coefficient $c_m^{(k\neq i)}$ then is

$$\left(\frac{\eta_m'}{\rho}+\eta_m''\right)(\mathrm{R}_m^{(i)}(\rho)+c_m^{(k\neq i)}\mathrm{R}_m^{(k\neq i)}(\rho))+2\eta_m'(\mathrm{R}_m'^{(i)}(\rho)+c_m^{(k\neq i)}\mathrm{R}_m'^{(k\neq i)}(\rho))\stackrel{!}{=}0. \quad (49)$$

A suitable equation for the coefficients can be obtained by multiplication by $\mathrm{R}_m^{(i)}(\rho)$ and integration. This procedure should give an *intrinsic mode* as defined by Arnold/Felsen [2] and Topuz and Felsen [3], which satisfies the wave equation (approximately), the boundary conditions and which contains the reflections produced by the wedge shape. The advantage over the construction of those authors seems to be that the requirement $k_0 r \gg 1$ must not be posed.

9. CONCLUDING REMARKS

Two methods were described for how wedge modes can be constructed. They are elementary solutions in a wedge-shaped space which obey the boundary conditions and the wave equation—at least approximately. Because they are orthogonal to each other over the wedge angle, they are suited for a modal analysis of practical tasks is such spaces. The conflict between the r-dependent characteristic equation and the wave equation is solved by a treatment of the "rest" of the wave equation.

A principally different procedure consists in taking elementary solutions which solve the wave equation exactly and solving the boundary condition with a superposition of such solutions. Suitable elementary solutions are the "ideal" modes

$$\varphi_m(\rho,\vartheta)=\cos(\varepsilon_m\vartheta)\mathrm{Z}_{\varepsilon_m}(\rho), \qquad \varepsilon_m=m\pi/\vartheta_0, \qquad m=0,1,2,,\ldots, \quad (50)$$

for both wedge boundaries rigid, and

$$\psi_m(\rho,\vartheta)=\cos(\eta_m\vartheta)\mathrm{Z}_{\eta_m}(\rho), \qquad \eta_m=(m+1/2)\pi/\vartheta_0, \qquad m=0,1,2,\ldots, \quad (51)$$

for the boundary at $\vartheta = 0$ rigid and the boundary at $\vartheta = \vartheta_0$ soft. The radial factors $Z_x(\rho)$ therein are cylindrical functions (Bessel, Neumann, Hankel functions or linear combinations of them). These ideal modes solve the wave equation exactly and satisfy the boundary condition at $\vartheta = 0$ as well as the Sommerfeld condition, and they are mutually orthogonal in $(0, \vartheta_0)$. The field in the wedge with an absorbing boundary is composed of

$$p(\rho, \vartheta) = \sum_{m \geqslant 0} A_m \varphi_m(\rho, \vartheta) + B_m \psi_m(\rho, \vartheta). \tag{52}$$

The boundary condition at the absorbing boundary then gives

$$\sum_{m \geqslant 0} (-1)^m (m + 1/2)\pi B_m Z_{\eta_m}(\rho) \stackrel{!}{=} \mathrm{j}\vartheta_0 G_0 \rho \sum_{m \geqslant 0} (-1)^m A_m Z_{\varepsilon_m}(\rho). \tag{53}$$

This, in ρ continuous, equation must be solved for A_m, B_m, both const (ρ). This can be done numerically either by a collocation technique or by a Galerkin procedure.

Although this procedure has some numerical appeal, as compared to the methods described above, it does not generate wedge modes in a lined wedge. It is not clear, further, if the set of the ideal modes is complete enough for the field synthesis in a lined wedge; the convergence of equation (52) to a surface wave mode, for example, will be slow.

The present paper describes methods for the construction of modes with which a modal analysis of sound fields in wedge-shaped spaces with lined flanks can be performed. Like similar theories for underwater acoustic applications the method has numerical procedures embedded in it. These are here the determination of the lateral wave numbers of the modes from the characteristic equation (this subtask seems to be unavoidable), the polynomial approximation of these solutions over the radial range, and the determination of the coefficients for the development of the "rest" (this step is not necessary for small wedge angle values). The advantages of the present method over known methods are seen to be (1) in the composition of the modes by known (hypergeometric) functions, (2) in the fact that the condition $k_0 r \gg 1$ is not necessary, and (3) in the expansion of the range of wedge angles. However, this expansion is not large enough to cover all wedge angles which may appear in the field of silencer ducts. Therefore, a subsequent paper will describe a method of larger wedge angles.

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APPENDIX: SOLUTION OF THE CHARACTERISTIC EQUATION

First, consider the characteristic equation (8) in the form

$$z \tan z = \mathrm{j}B \tag{A1}$$

for symmetrical modes with the wanted quantity $z = \eta\vartheta_0$ and the given quantity $B = \rho\vartheta_0 G_0$. One needs a set of solutions z_n with running ρ in B. Apply Muller's method for the numerical solution of transcendental equations, because it is rather robust against jumping between solutions. Muller's method needs three start values $z_n^{(s)}$, $s = 1, 2, 3$, but it does not need the derivative $f'(z)$ of the function $f(z)$ in the equation $f(z) = 0$. When the solutions z_n are well separated, it is sufficient to have available one approximate solution $z_n^{(1)}$ and to take for the other $z_n^{(s)}$ values in the neighbourhood of $z_n^{(1)}$. The solutions are well separated for small $|B|$, i.e., at low ρ, and for large $|B|$, i.e., at high ρ. When one begins the iteration through ρ_1 (iteration index $i = 1, 2, \ldots$, the corresponding solutions $z_{n,i}$) with a small ρ value, then one can take $z_{n,1}^{(1)} = n\pi$, $z_{n,1}^{(2)} = n\pi + 0{\cdot}01$, $z_{n,1}^{(3)} = n\pi + 0{\cdot}01 + 0{\cdot}01\mathrm{j}$. For the next steps of iteration, one needs more precise start values. With the expansion of $\tan z$ in a continued fraction at $z = n\pi$ one gets, from the characteristic equation,

$$z_n^2 \approx n\pi z + \mathrm{j}B\left[1 - \frac{(z-n\pi)^2}{3-}\,\frac{(z-n\pi)^2}{5-\cdots}\cdots\frac{(z-n\pi)^2}{(n_{hi}-2)^2-(z-n\pi)^2/n_{hi}^2}\right]$$

$$\xrightarrow[z\to m\pi]{} (n\pi)^2 + \mathrm{j}B \xrightarrow[\rho\to 0 \text{ and/or } G_0\to 0]{} (n\pi)^2. \tag{A2}$$

So one takes as starters $z_{n,i}^{(2)}$ in higher iteration steps $z_{n,i}^{(1)} = z_{n,i-1}$, i.e., the solution from the previous step; for $z_{n,i}^{(3)}$ one takes the square root (with positive real component) of the first line of equation (A2) with $z \to z_{n,i-1}$ and B computed for ρ_i; and finally, one takes $z_{n,i}^{(2)} = (z_{n,i}^{(1)} + z_{n,i}^{(3)})/2$. If $z_{n,i}$ is applied in equation (A2), this would be the continued fraction expansion of the characteristic equation. Its convergence is fastest for the solution z_n, and there it is slowest if this solution belongs to a surface wave mode. A value $n_{hi} = 11$ for the termination of the continued wave expansion also takes this case into consideration. The role of $z_{n,i}^{(3)}$ from equation (A2) is to direct the numerical procedure of Muller's method from the previous solution $z_{n,i-1}$ into the right direction of the wanted solution $z_{n,i}$.

The case of antisymmetrical modes (soft boundary at $\vartheta = 0$) is treated similarly. They have the characteristic equation $z \cot(z) = -\mathrm{j}B$. Because $\cot(z) = -\tan(z - (n + 1/2)\pi)$, one has to replace $n \to (n + 1/2)$ in equation (A2).

If one would like to begin the iteration through ρ from high ρ values for which $|B| \gg 1$, suitable first starters would be $z_{n,1}^{(1)} = (n \pm 1/2)\pi$, $z_{n,1}^{(2)} = (n \pm 1/2)\pi \mp 0{\cdot}01$, $z_{n,1}^{(3)} = (n \pm 1/2)\pi \mp 0{\cdot}01 + 0{\cdot}01\mathrm{j}$ in which the signs correspond to $n \leqslant n_s$ or $n > n_s$, respectively, where n_s is the mode order of the surface wave mode if it exists in the case Im $\{G_0\} > 0$. It is determined from the point of intersection of the straight line through (0, B) with the curve interconnecting the branch points $B_{b,n}$; its value is n_s when the straight line passes the interconnection curve between $B_{b,ns}$ and $B_{b,ns+1}$. Approximate equations (sufficiently precise for the present purpose, taken from reference [1, chapter 26]) of the interconnection curves are (with $B_b = B_b' + \mathrm{j}B_b''$)

$$\begin{aligned} B''_{b,n} &= 2{\cdot}02599(B'_{b,n})^{1/3} - 0{\cdot}655631(B'_{b,n})^{1/2} \\ &\quad + 0{\cdot}00631631B'_{b,n} + 0{\cdot}00000250827B'^2_{b,n}, \quad \text{symmetrical}, \\ B''_{b,n} &= 1{\cdot}0 + 0{\cdot}553673(B'_{b,n})^{1/2} \\ &\quad - 0{\cdot}0412894B'_{b,n} + 0{\cdot}000120216B'^2_{b,n}, \quad \text{antisymmetrical}, \end{aligned} \tag{A3}$$

$$\begin{aligned} B'_b(n) &= -1{\cdot}50237\sqrt{n} + 3{\cdot}76029n \\ &\quad - 0{\cdot}0284415n^2 + 0{\cdot}000620241n^3, \quad \text{symmetrical}, \\ B'_b(n) &= 3{\cdot}39042n - 0{\cdot}023312n^2 + 0{\cdot}000651624n^3, \quad \text{antisymmetrical}. \end{aligned} \tag{A4}$$

The foregoing determination of n_s is not necessary when one begins the iteration with small ρ values.

The precision of the coefficients s_i in the polynomial representation (see equation (11)),

$$\mathrm{j}\rho G'_\vartheta(\vartheta_0) = \eta^2 - (\rho G_0)^2 \approx s_0 + s_1,\, \rho + s_2\rho^2, \tag{A5}$$

is better if they are determined separately for the real and imaginary components of $\eta^2 - (\rho G_0)^2$.